Boundary Layer for a Non-Newtonian Flow over a Rough Surface

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1 Introduction

The general concern of this paper is the effect of rough walls on fluids. This effect is important at various scales. For instance, in the area of microfluidics, recent experimental works have emphasized the role of hydrophobic rough walls in the improvement of slipping properties of microchannels. Also, in geophysics, as far as large scale motions are concerned, topography or shore variations can be assimilated to roughness. For high Reynolds number flows, an important issue is to understand how localized roughness triggers instabilities, and transition to turbulence. For laminar flows, the point is rather to understand how distributed roughness may have a macroscopic impact on the dynamics. More precisely, the hope is to be able to encode an averaged effect through an effective boundary condition at a smoothened wall. Such boundary condition, called *a wall law*, will avoid to simulate the small-scale dynamics that takes place in a boundary layer in the vicinity of the rough surface.

The derivation of wall laws for laminar Newtonian flows has been much studied, since the pioneering works of Achdou, Pironneau and Valentin [1, 2], or Jäger and Mikelić [19, 20]. See also [23, 3, 15, 8, 25]. A natural mathematical approach of this problem is by homogenization techniques, the roughness being modeled by a small amplitude/high frequency oscillation. Typically, one considers a Navier-Stokes flow in a channel Ω^ε with a rough bottom:

$$\Omega^\varepsilon = \Omega \cup \Sigma_0 \cup R^\varepsilon.$$

Precisely:

- $\Omega = (0,1)^2$ is the flat portion of the channel.
- R^ε is the rough portion of the channel: it reads

$$R^\varepsilon = \{x = (x_1, x_2), x_1 \in (0,1), 0 > x_2 > \varepsilon\gamma(x_1/\varepsilon)\}$$

  with a bottom surface $\Gamma^\varepsilon := \{x_2 = \varepsilon\gamma(x_1/\varepsilon)\}$ parametrized by $\varepsilon \ll 1$. Function $\gamma = \gamma(y_1)$ is the *roughness pattern*.
- Eventually, $\Sigma_0 := (0,1) \times \{0\}$ is the interface between the rough and flat part. It is the artificial boundary at which the wall law is set.

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Of course, within such model, the goal is to understand the asymptotic behavior of the Navier-Stokes solution u^ε as $\varepsilon \to 0$. Therefore, the starting point is a formal approximation of u^ε under the form

$$u^\varepsilon_{app}(x) = u^0(x) + \varepsilon u^1(x) + \cdots + u^0_{bl}(x/\varepsilon) + \varepsilon u^1_{bl}(x/\varepsilon) + \ldots. \tag{1}$$

In this expansion, the $u^i = u^i(x)$ describe the large-scale part of the flow, whereas the $u^i_{bl} = u^i_{bl}(y)$ describe the boundary layer. The typical variable $y = x/\varepsilon$ matches the small-scale variations induced by the roughness. In the case of homogeneous Dirichlet conditions at Γ^ε, one can check formally that:

- u^0 is the solution of the Navier-Stokes equation in Ω, with Dirichlet condition at Σ_0.
- $u^0_{bl} = 0$, whereas u^1_{bl} satisfies a Stokes equation in variable y in the boundary layer domain

$$\Omega_{bl} := \{y = (y_1, y_2), y_1 \in \mathbb{R}, y_2 > \gamma(y_1)\}.$$

The next step is to solve this boundary layer system, and show convergence of u^1_{bl} as $y_2 \to +\infty$ to a constant field $u^\infty = (U^\infty, 0)$. This in turn determines the appropriate boundary condition for the large scale correction u^1. From there, considering the large scale part $u^0 + \varepsilon u^1$, one can show that:

- The limit wall law is a homogeneous Dirichlet condition. Let us point out that this feature persists even starting from a microscopic pure slip condition, under some non-degeneracy of the roughness: [9, 5, 6].
- The $O(\varepsilon)$ correction to this wall law is a slip condition of Navier type, with $O(\varepsilon)$ slip length.

All these steps were completed in aforementioned articles, in the case of periodic roughness pattern γ : $\gamma(y_1 + 1) = \gamma(y_1)$. Over the last years, the first author has extended this analysis to general patterns of roughness, with ergodicity properties (random stationary distribution of roughness, *etc*). We refer to [4, 16, 17]. See also [12] for some recent work on the same topic.

The purpose of the present paper is to extend the former analysis to non-Newtonian flows. This may have various sources of interest. One can think of engineering applications, for instance lubricants to which polymeric additives confer a shear thinning behavior. One can also think of glaciology: as the interaction of glaciers with the underlying rocks is unavailable, wall laws can help. From a mathematical point of view, such examples may be described by a power-law model. Hence, we consider a system of the following form:

$$\begin{cases} -\mathrm{div}\, S(Du) + \nabla p = e_1 & \text{in } \Omega^\varepsilon, \\ \mathrm{div}\, u = 0 & \text{in } \Omega^\varepsilon, \\ u|_{\Gamma^\varepsilon} = 0, \quad u|_{x_2=1} = 0, & u \text{ 1-periodic in } x_1. \end{cases} \tag{2}$$

As usual, $u = u(x) \in \mathbb{R}^2$ is the velocity field, $p = p(x) \in \mathbb{R}$ is the pressure. The source term e_1 at the right-hand side of the first equation corresponds to a constant pressure gradient $e_1 = (1, 0)^t$ throughout the channel. Eventually, the left-hand side involves the stress tensor of the fluid. As mentioned above, it is taken of power-law type: $S : \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}^{2\times 2}_{\mathrm{sym}}$ is given by

$$S : \mathbb{R}^{2\times 2}_{\mathrm{sym}} \to \mathbb{R}^{2\times 2}_{\mathrm{sym}}, \quad S(A) = \nu |A|^{p-2} A, \quad \nu > 0, \quad 1 < p < +\infty, \tag{3}$$

where $|A| = (\sum_{i,j} a^2_{i,j})^{1/2}$ is the usual euclidean norm of the matrix A. *For simplicity, we shall take* $\nu = 1$. Hence, $S(Du) = |Du|^{p-2} Du$, where we recall that $Du = \frac{1}{2}(\nabla u + (\nabla u)^t)$ is the symmetric part of the jacobian. Following classical terminology, the case $p < 2$ resp. $p > 2$ corresponds to *shear thinning* fluids, resp. *shear thickening* fluids. The limit case $p = 2$ describes a Newtonian flow. Note that we complete the equation in system (2) by a standard no-slip condition at the top and bottom boundary of the channel. For the sake of simplicity, we assume periodicity in the large scale horizontal variable x_1. Finally, we also make a simplifying periodicity assumption on the roughness pattern γ:

$$\gamma \text{ is } C^{2,\alpha} \text{ for some } \alpha > 0, \text{ has values in } (-1, 0), \text{ and is 1-periodic in } y_1. \tag{4}$$

For every $\varepsilon > 0$ and any value of p, the generalized Stokes system (2) has a unique solution

$$(u^\varepsilon, p^\varepsilon) \in W^{1,p}(\Omega^\varepsilon) \times L^{p'}(\Omega^\varepsilon)/\mathbb{R}.$$

The main point is to know about the asymptotic behavior of u^ε, precisely to build some good approximate solution. With regards to the Newtonian case, we anticipate that this approximation will take a form close to (1). Our plan is then:

- to derive the equations satisfied by the first terms of expansion (1).
- to solve these equations, and show convergence of the boundary layer term to a constant field away from the boundary.
- to obtain error estimates for the difference between u^ε and u^ε_{app}.
- to derive from there appropriate wall laws.

This program will be more difficult to achieve for non-Newtonian fluids, in particular for the shear thinning case $p < 2$, notably as regards the study of the boundary layer equations on $u_{bl} := u^1_{bl}$. In short, these equations will be seen to read

$$-\mathrm{div}\,(S(A + Du_{bl})) + \nabla p = 0, \mathrm{div}\, u = 0, \quad y \in \Omega_{bl}$$

for some explicit matrix A, together with periodicity condition in y_1 and a homogeneous Dirichlet condition at the bottom of Ω_{bl}. Due to the nonlinearity of these equations and the fact that $A \neq 0$, the analysis will be much more difficult than in the Newtonian case, notably the proof of the so-called Saint-Venant estimates. We refer to section 2.2 for all details.

Let us conclude this introduction by giving some references on related problems. In [24]; E. Marušić-Paloka considers power-law fluids with convective terms in infinite channels and pipes (the non-Newtonian analogue of the celebrated Leray's problem). After an appropriate change of unknown, the system studied in [24] bears some strong similarity to our boundary layer system. However, it is different at two levels : first, the analysis is restricted to the case $p > 2$. Second, our lateral periodicity condition in y_1 is replaced by a no-slip condition. This allows to use Poincaré's inequality in the transverse variable, and control zero order terms (in velocity u) by ∇u, and then by Du through the Korn inequality. It simplifies in this way the derivation of exponential convergence of the boundary layer solution (Saint-Venant estimates). The same simplification holds in the context of paper [7], where the behaviour of a Carreau flow through a thin filter is analysed. The corrector describing the behaviour of the fluid near the filter is governed by a kind of boundary layer type system, in a slab that is infinite vertically in both directions. In this setting, one has $A = 0$, and the authors refer to [24] for well-posedness and qualitative behaviour. We also refer to the recent article [26] dedicated to power-law fluids in thin domains, with Navier condition and anisotropic roughness (with a wavelength that is larger than the amplitude). In this setting, no boundary layer analysis is needed, and the author succeeds to describe the limit asymptotics by the unfolding method. Finally, we point out the very recent paper [11], where an Oldroyd fluid is considered in a rough channel. In this setting, no nonlinearity is associated to the boundary layer, which satisfies a Stokes problem.

2 Boundary layer analysis

From the Newtonian case, we expect the solution $(u^\varepsilon, p^\varepsilon)$ of (2) to be approximated by

$$u^\varepsilon \approx u^0(x) + \varepsilon u_{bl}(x/\varepsilon), \quad p^\varepsilon \approx p^0(x) + p_{bl}(x/\varepsilon),$$

where

- (u^0, p^0) describes the flow away from the boundary layer. We shall take $u^0 = 0$ and $p^0 = 0$ in the rough part R^ε of the channel.

- $(u_{bl}, p_{bl}) = (u_{bl}, p_{bl})(y)$ is a boundary layer corrector defined on the slab

  $$\Omega_{bl} := \{y = (y_1, y_2), y_1 \in \mathbb{T}, y_2 > \gamma(y_1)\},$$

  where $\mathbb{T}$ is the torus $\mathbb{R}/\mathbb{Z}$. This torus corresponds implicitly to a periodic boundary condition in y_1, which is inherited from the periodicity of the roughness pattern γ. We denote

  $$\Omega_{bl}^{\pm} := \Omega_{bl} \cap \{\pm y_2 > 0\}$$

  its upper and lower parts, and

  $$\Gamma_{bl} := \{y = (y_1, y_2), y_1 \in \mathbb{T}, y_2 = \gamma(y_1)\}$$

  its bottom boundary. As the boundary layer corrector is supposed to be localized, we expect that

  $$\nabla u_{bl} \to 0 \quad \text{as } y_2 \to +\infty.$$

With this constraint in mind, we take (u^0, p^0) to be the solution of

$$\begin{cases} -\mathrm{div}\, S(Du^0) + \nabla p^0 = e_1 & \text{in } \Omega, \\ \mathrm{div}\, u^0 = 0 & \text{in } \Omega, \\ u^0|_{\Sigma_0} = 0, \quad u^0|_{x_2=1} = 0, & u^0 \text{ 1-periodic in } x_1. \end{cases} \tag{5}$$

The solution is explicit and generalizes the Poiseuille flow. A simple calculation yields: for all $x \in \Omega$,

$$p^0(x) = 0, \quad u^0(x) = (U(x_2), 0), \quad U(x_2) = \frac{p-1}{p}\left(\sqrt{2}^{-\frac{p}{(p-1)}} - \sqrt{2}^{\frac{p}{(p-1)}}\left|x_2 - \frac{1}{2}\right|^{\frac{p}{p-1}}\right). \tag{6}$$

We extend this solution to the whole rough channel by taking: $u^0 = 0, p^0 = 0$ in R^ε. This zero order approximation is clearly continuous across the interface Σ_0, but the associated stress is not: denoting

$$A := D(u^0)|_{y_2=0^+} = \frac{1}{2}\begin{pmatrix} 0 & U'(0) \\ U'(0) & 0 \end{pmatrix}, \quad \text{with } U'(0) = \sqrt{2}^{\frac{p-2}{p-1}} \tag{7}$$

we obtain

$$[S(Du^0)n - p^0 n]|_{\Sigma_0} = |A|^{p-2}An = \begin{pmatrix} -\sqrt{2}^{-p}U'(0)^{p-1} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ 0 \end{pmatrix}$$

with $n = -e_2 = -(0, 1)^t$ and $[f] := f|_{x_2=0^+} - f|_{x_2=0^-}$.

This jump should be corrected by u_{bl}, so that the total approximation $u^0(x) + \varepsilon u_{bl}(x/\varepsilon)$ has no jump. This explains the amplitude ε of the boundary layer term, as its gradient will then be $O(1)$. By Taylor expansion $U(x_2) = U(\varepsilon y_2) = U(0) + \varepsilon U'(0)y_2 + \ldots$ we get formally $D(u^0 + \varepsilon u_{bl}(\cdot/\varepsilon)) \approx A + Du_{bl}$, where the last symmetric gradient is with respect to variable y. We then derive the following boundary layer system:

$$\begin{cases} -\mathrm{div}\, S(A + Du_{bl}) + \nabla p_{bl} = 0 & \text{in } \Omega_{bl}^+, \\ -\mathrm{div}\, S(Du_{bl}) + \nabla p_{bl} = 0 & \text{in } \Omega_{bl}^-, \\ \mathrm{div}\, u_{bl} = 0 & \text{in } \Omega_{bl}^+ \cup \Omega_{bl}^-, \\ u_{bl}|_{\Gamma_{bl}} = 0, & \\ u_{bl}|_{y_2=0^+} - u_{bl}|_{y_2=0^-} = 0, & \end{cases} \tag{8}$$

together with the jump condition

$$(S(A + Du_{bl})n - p_{bl}n)\,|_{y_2=0^+} - (S(Du_{bl})n - p_{bl}n)\,|_{y_2=0^-} = 0, \quad n = (0, -1)^t. \tag{9}$$

Let us recall that the periodic boundary condition in y_1 is encoded in the definition of the boundary layer domain. The rest of this section will be devoted to the well-posedness and qualitative properties of (8)-(9). We shall give detailed proofs only for the more difficult case $p < 2$, and comment briefly on the case $p \geq 2$ at the end of the section. Our main results will be the following:

Theorem 2.1 (Well-posedness)
For all $1 < p < 2$, (8)-(9) has a unique solution $(u_{bl}, p_{bl}) \in W^{1,p}_{loc}(\Omega_{bl}) \times L^{p'}_{loc}(\Omega_{bl})/\mathbb{R}$ satisfying for any $M > |A|$:

$$Du_{bl}\, 1_{\{|Du_{bl}| \le M\}} \in L^2(\Omega_{bl}), \quad Du_{bl}\, 1_{\{|Du_{bl}| \ge M\}} \in L^p(\Omega_{bl}).$$

For all $p \ge 2$, (8)-(9) has a unique solution $(u_{bl}, p_{bl}) \in W^{1,p}_{loc}(\Omega_{bl}) \times L^{p'}_{loc}(\Omega_{bl})/\mathbb{R}$ s.t. $Du_{bl} \in L^p(\Omega_{bl}) \cap L^2(\Omega_{bl})$.

Theorem 2.2 (Exponential convergence)
For any $1 < p < +\infty$, the solution given by the previous theorem converges exponentially, in the sense that for some $C, \delta > 0$

$$|u_{bl}(y) - u^\infty| \le C e^{-\delta y_2} \quad \forall\, y \in \Omega^+_{bl},$$

where $u^\infty = (U^\infty, 0)$ is some constant horizontal vector field.

2.1 Well-posedness

A priori estimates

We focus on the case $1 < p < 2$, and provide the *a priori* estimates on which the well-posedness is based. The easier case $p \ge 2$ is discussed at the end of the paragraph. As A is a constant matrix, we have from (8):

$$-\mathrm{div}\, S(A + Du_{bl}) + \mathrm{div}\, S(A) + \nabla p_{bl} = 0 \quad \text{in } \Omega^+_{bl}, \quad -\mathrm{div}\, S(Du_{bl}) + \nabla p_{bl} = 0 \quad \text{in } \Omega^-_{bl}.$$

We multiply the two equations by Du_{bl} and integrate over Ω^+_{bl} and Ω^-_{bl} respectively. After integrations by parts, accounting for the jump conditions at $y_2 = 0$, we get

$$\int_{\Omega^+_{bl}} (S(A + Du_{bl}) - S(A)) : Du_{bl}\, \mathrm{d}y + \int_{\Omega^-_{bl}} S(Du_{bl}) : Du_{bl}\, \mathrm{d}y = -\int_{y_2=0} S(A)n \cdot u_{bl}\, \mathrm{d}S. \tag{10}$$

The right-hand side is controlled using successively Poincaré and Korn inequalities (for the Korn inequality, see the appendix):

$$|\int_{y_2=0} S(A)n \cdot u_{bl}\, \mathrm{d}y| \le C\|u_{bl}\|_{L^p(\{y_2=0\})} \le C'\|\nabla u_{bl}\|_{L^p(\Omega^-_{bl})} \le C''\|Du_{bl}\|_{L^p(\Omega^-_{bl})}. \tag{11}$$

As regards the left-hand side, we rely on the following vector inequality, established in [22, p74, eq. (VII)]: for all $1 < p \le 2$, for all vectors a, b

$$(|b|^{p-2}b - |a|^{p-2}a \mid b - a) \;\ge\; (p-1)|b-a|^2 \int_0^1 |a + t(b-a)|^{p-2} dt. \tag{12}$$

In particular, for any $M > 0$, if $|b - a| \le M$, one has

$$(|b|^{p-2}b - |a|^{p-2}a \mid b - a) \;\ge\; \frac{p-1}{(|a| + M)^{2-p}} |b-a|^2, \tag{13}$$

whereas if $|b - a| > M > |a|$, we get

$$(|b|^{p-2}b - |a|^{p-2}a \mid b - a) \;\ge (p-1)|b-a|^2 \int_{\frac{|a|}{|b-a|}}^1 (2t|b-a|)^{p-2}\, dt \ge 2^{p-3}\left(1 - (|a|/M)^{p-1}\right)|b-a|^p. \tag{14}$$

We then apply such inequalities to (10), taking $a = A$, $b = A + Du_{bl}$. For $M > |A|$, there exists c dependent on M such that

$$\int_{\Omega^+_{bl}} (S(A + Du_{bl}) - S(A)) : Du_{bl}\, \mathrm{d}y \ge c \int_{\Omega^+_{bl}} 1\!\!1_{\{|Du_{bl}| \le M\}} |Du_{bl}|^2\, \mathrm{d}y \;+\; \int_{\Omega^+_{bl}} 1\!\!1_{\{|Du_{bl}| > M\}} |Du_{bl}|^p\, \mathrm{d}y,$$

so that for some C dependent on M

$$\int_{\Omega_{bl}^+} |\mathbb{1}_{\{|Du_{bl}|\le M\}} Du_{bl}|^2 \, \mathrm{d}y \; + \; \int_{\Omega_{bl}^+} |\mathbb{1}_{\{|Du_{bl}|>M\}} Du_{bl}|^p \, \mathrm{d}y + \int_{\Omega_{bl}^-} |Du_{bl}|^p \, \mathrm{d}y \; \le \; C \, \|Du_{bl}\|_{L^p(\Omega_{bl}^-)}.$$

Hence, still for some C dependent on M:

$$\int_{\Omega_{bl}^+} 1_{\{|Du_{bl}|\le M\}} |Du_{bl}|^2 \, \mathrm{d}y \; + \; \int_{\Omega_{bl}^+} 1_{\{|Du_{bl}|>M\}} |Du_{bl}|^p \, \mathrm{d}y + \int_{\Omega_{bl}^-} |Du_{bl}|^p \, \mathrm{d}y \; \le \; C. \tag{15}$$

This is the *a priori* estimate on which Theorem 2.1 can be established (for $p \in]1,2]$). Note that this inequality implies that for any height h,

$$\|Du_{bl}\|_{L^p(\Omega_{bl}\cap\{y_2\le h\})} \le C_h$$

(bounding the L^p norm by the L^2 norm on a bounded set). Combining with Poincaré and Korn inequalities, we obtain that u_{bl} belongs to $W^{1,p}_{loc}(\Omega_{bl})$.

In the case $p \ge 2$, we can directly use the following inequality, which holds for all $a, \; b \in \mathbb{R}^n$:

$$|a-b|^p 2^{2-p} \le 2^{-1} \left(|b|^{p-2} + |a|^{p-2}\right) |b-a|^2 \le \left\langle |a|^{p-2}a - |b|^{p-2}|b|, a-b \right\rangle. \tag{16}$$

It provides both an L^2 and L^p control of the symmetric gradient of the solution. Indeed, taking $a = A + D_y u_{bl}$, $b = A$ and using (10) we get the following a'priori estimates for $p \ge 2$

$$\begin{aligned}
&2^{2-p} \int_{\Omega_{bl}^+} |Du_{bl}|^p \, \mathrm{d}y + \int_{\Omega_{bl}^-} |Du_{bl}|^p \, \mathrm{d}y + 2^{-1}|A|^{p-2} \int_{\Omega_{bl}^+} |Du_{bl}|^2 \, \mathrm{d}y \\
&\le \int_{\Omega_{bl}^+} (S(A+Du_{bl}) - S(A)) : Du_{bl} \, \mathrm{d}y + \int_{\Omega_{bl}^-} S(Du_{bl}) : Du_{bl} \, \mathrm{d}y \\
&= \; - \int_{\Sigma_0} S(A) n \cdot u_{bl} \, \mathrm{d}S \\
&\le \; c(\alpha) \|S(A)\|^{p'}_{L^{p'}(\Sigma_0)} + \alpha \|u_{bl}\|^p_{L^p(\Sigma_0)} \le \; c(\alpha) \|S(A)\|^{p'}_{L^{p'}(\Sigma_0)} + \alpha C_\Gamma \|\nabla u_{bl}\|^p_{L^p(\Omega_{bl}^-)} \\
&\le c(\alpha) \|S(A)\|^{p'}_{L^{p'}(\Sigma_0)} + \alpha C_\Gamma C_K \|Du_{bl}\|^p_{L^p(\Omega_{bl}^-)},
\end{aligned} \tag{17}$$

where the trace theorem, the Poincaré inequality and the Korn inequality were employed. By choosing the coefficient α small enough, and by the imbedding of $L^p(\Omega_{bl}^-)$ in $L^2(\Omega_{bl}^-)$, (17) provides

$$\int_{\Omega_{bl}} |Du_{bl}|^p + |Du_{bl}|^2 \, \mathrm{d}y \le C \|S(A)\|^{p'}_{L^{p'}(\Sigma_0)} < \infty. \tag{18}$$

Eventually, by Korn and Poincaré inequalities: $u_{bl} \in W^{1,p}(\Omega_{bl})$ for $2 \le p < \infty$.

Construction scheme for the solution

We briefly explain how to construct a solution satisfying the above estimates. We restrict to the most difficult case $p \in]1,2]$. There are two steps:

Step 1: we solve the same equations, but *in the bounded domain* $\Omega_{bl,n} = \Omega_{bl} \cap \{y_2 < n\}$, with a Dirichlet boundary condition at the top. As $\Omega_{bl,n}$ is bounded, the imbedding of $W^{1,p}$ in L^p is compact, so that a solution $u_{bl,n}$ can be built in a standard way. Namely, one can construct a sequence of Galerkin approximations $u_{bl,n,m}$ by Schauder's fixed point theorem. Then, as the estimate (15) holds for $u_{bl,n,m}$ uniformly in m and n, the sequence $Du_{bl,n,m}$ is bounded in $L^p(\Omega_{bl,n})$ uniformly in m. Sending m to infinity yields a solution $u_{bl,n}$, the convergence of the nonlinear stress tensor follows from Minty's trick. Note that one can then perform on $u_{bl,n}$ the manipulations of the previous paragraph, so that it satisfies (15) uniformly in n.

Step 2: we let n go to infinity. We first extend $u_{bl,n}$ by 0 for $y_2 > n$, and fix $M > |A|$. From the uniform estimate (15), we get easily the following convergences (up to a subsequence):

$$\begin{aligned}
&u_{bl,n} \to u_{bl} \text{ weakly in } W^{1,p}_{loc}(\Omega_{bl}),\\
&Du_{bl,n} \to Du_{bl} \text{ weakly in } L^p(\Omega^-_{bl}),\\
&Du_{bl,n}\mathbb{1}_{|Du_{bl,n}|<M} \to V_1 \text{ weakly in } L^2(\Omega^+_{bl}), \quad \text{weakly-* in } L^\infty(\Omega^+_{bl}),\\
&Du_{bl,n}\mathbb{1}_{|Du_{bl,n}|\ge M} \to V_2 \text{ weakly in } L^p(\Omega^+_{bl}).
\end{aligned} \tag{19}$$

Of course, $Du_{bl} = V_1 + V_2$ in Ω^+_{bl}. A key point is that

$$S(A + Du_{bl,n}) - S(A) \text{ is bounded uniformly in } n \text{ in } (L^p(\Omega^+_{bl}))' = L^{p'}(\Omega^+_{bl}) \text{ and in } \left(L^2(\Omega^+_{bl}) \cap L^\infty(\Omega^+_{bl})\right)'.$$

and converges weakly-* to some S^+ in that space. To establish this uniform bound, we treat separately

$$S_{n,1} \;:=\; (S(A + Du_{bl,n}) - S(A))\mathbb{1}_{|Du_{bl,n}|<M}, \quad S_{n,2} \;:=\; (S(A + Du_{bl,n}) - S(A))\mathbb{1}_{|Du_{bl,n}|\ge M}.$$

- For $S_{n,1}$, we use the inequality (26). It gives $|S_{n,1}| \le C|Du_{bl,n}|\mathbb{1}_{|Du_{bl,n}|<M}$, which provides a uniform bound in $L^2 \cap L^\infty$, and so in particular in $L^{p'}$ and in L^2.
- For $S_{n,2}$, we use first that $|S_{n,2}| \le C|Du_{bl,n}|^{p-1}\mathbb{1}_{|Du_{bl,n}|\ge M}$, so that it is uniformly bounded in $L^{p'}$. We use then (26), so that $|S_{n,2}| \le C|Du_{bl,n}|\mathbb{1}_{|Du_{bl,n}|\ge M}$, which yields a uniform bound in L^p, in particular in $(L^2 \cap L^\infty)'$ ($p \in]1, 2]$).

From there, standard manipulations give

$$\int_{\Omega^+_{bl}} (S(A + Du_{bl,n}) - S(A)) : Du_{bl,n} \to \int_{\Omega^+_{bl}} S^+ : (V_1 + V_2) = \int_{\Omega^+_{bl}} S^+ : Du_{bl}$$

One has even more directly

$$\int_{\Omega^-_{bl}} S(Du_{bl,n}) : Du_{bl,n} \to \int_{\Omega^-_{bl}} S^- : Du_{bl}$$

and one concludes by Minty's trick that $S^+ = S(A+Du_{bl})-S(A)$, $S^- = S(Du_{bl})$. It follows that u_{bl} satisfies (8)-(9) in a weak sense. Finally, one can perform on u_{bl} the manipulations of the previous paragraph, so that it satisfies (15).

Uniqueness

Let u^1_{bl} and u^2_{bl} be weak solutions of (8)-(9), that is satisfying the variational formulation

$$\int_{\Omega^+_{bl}} S(A + Du^i_{bl}) : D\varphi \;+\; \int_{\Omega^-_{bl}} S(Du^i_{bl}) : D\varphi = -\int_{y_2=0} S(A)n \cdot \varphi \,\mathrm{d}S, \quad i = 1, 2 \tag{20}$$

for all smooth divergence free fields $\varphi \in C^\infty_c(\Omega_{bl})$. The point is then to replace φ by $u^1_{bl} - u^2_{bl}$, to obtain

$$\int_{\Omega^+_{bl}} \left(S(A + Du^1_{bl}) - S(A + Du^2_{bl})\right) : D(u^1_{bl} - u^2_{bl}) \;+\; \int_{\Omega^-_{bl}} (S(Du^1_{bl}) - S(Du^2_{bl}) : D(u^1_{bl} - u^2_{bl}) = 0. \tag{21}$$

Rigorously, one constructs by convolution a sequence φ_n such that $D\varphi_n$ converges appropriately to $Du^1_{bl} - Du^2_{bl}$. In the case $p < 2$, the convergence holds in $(L^2(\Omega^+_{bl}) \cap L^\infty(\Omega^+_{bl})) + L^p(\Omega^+_{bl})$, respectively in $L^p(\Omega^-_{bl})$. One can pass to the limit as n goes to infinity because

$$S(A + Du^1_{bl}) - S(A + Du^2_{bl}) = \left(S(A + Du^1_{bl}) - S(A)\right) + \left(S(A) - S(A + Du^2_{bl})\right),$$

respectively $S(Du^1_{bl}) - S(Du^2_{bl})$, belongs to the dual space: see the arguments of the construction scheme of section 2.1. Eventually, by strict convexity of $M \to |M|^p$ ($p > 1$), (21) implies that $Du^1_{bl} = Du^2_{bl}$. This implies that $u^1_{bl} - u^2_{bl}$ is a constant (dimension is 2), and due to the zero boundary condition at $\partial\Omega_{bl}$, we get $u^1_{bl} = u^2_{bl}$.

2.2 Saint-Venant estimate

We focus in this paragraph on the asymptotic behaviour of u_{bl} as y_2 goes to infinity. The point is to show exponential convergence of u_{bl} to a constant field. At first, we can use interior regularity results for the generalized Stokes equation in two dimensions. In particular, pondering on the results of [28] for $p < 2$, and [21] for $p \geq 2$, we have :

Lemma 2.1 *The solution built in Theorem 2.1 satisfies: u_{bl} has $C^{1,\alpha}$ regularity over $\Omega_{bl} \cap \{y_2 > 1\}$ for some $0 < \alpha < 1$. In particular, ∇u_{bl} is bounded uniformly over $\Omega_{bl} \cap \{y_2 > 1\}$.*

Proof. Let $0 \leq t < s$. We define $\Omega^{t,s}_{bl} := \Omega_{bl} \cap \{t < y_2 \leq s\}$. Note that $\Omega_{bl} \cap \{y_2 > 1\} = \cup_{t\in\mathbb{N}_*} \Omega^{t,t+1}_{bl}$. Moreover, from the *a priori estimate* (15) or (18), we deduce easily that

$$\|Du_{bl}\|_{L^p(\Omega^{t,t+2}_{bl})} \leq C \tag{22}$$

for all $t > 0$, for a constant C that does not depend on t. We then introduce:

$$v_t := 2A\left(0, y_2 - (t+\frac{1}{2})\right) + u_{bl} - \frac{1}{2}\int_{\Omega^{t-\frac{1}{2},t+\frac{3}{2}}_{bl}} u_{bl}\,\mathrm{d}y, \quad y \in \Omega^{t-\frac{1}{2},t+\frac{3}{2}}_{bl}, \quad \forall t \in \mathbb{N}_*.$$

From (8):

$$-\mathrm{div}\,(S(Dv_t)) + \nabla p_{bl} = 0, \quad \mathrm{div}\, v_t = 0 \quad \text{in } \Omega^{t-1/2,t+3/2}_{bl}, \quad \forall t \in \mathbb{N}_*.$$

Moreover, we get for some C independent of t:

$$\|v_t\|_{W^{1,p}(\Omega^{t-1/2,t+3/2}_{bl})} \leq C \quad \forall t \in \mathbb{N}_*. \tag{23}$$

Note that this $W^{1,p}$ control follows from (22): indeed, one can apply the Poincaré inequality for functions with zero mean, and then the Korn inequality. One can then ponder on the interior regularity results of articles [28] and [21], depending on the value of p: v_t has $C^{1,\alpha}$ regularity over $\Omega^{t,t+1}_{bl}$ for some $\alpha \in (0,1)$ (independent of t): for some C',

$$\|v_t\|_{C^{1,\alpha}(\Omega^{t,t+1}_{bl})} \leq C', \quad \text{and in particular} \quad \|\nabla v_t\|_{L^\infty(\Omega^{t,t+1}_{bl})} \leq C' \quad \forall t \in \mathbb{N}_*.$$

Going back to u_{bl} concludes the proof of the lemma.

We are now ready to establish a keypoint in the proof of Theorem 2.2, called a *Saint-Venant estimate*: namely, we show that the energy of the solution located above $y_2 = t$ decays exponentially with t. In our context, a good energy is

$$E(t) := \int_{\{y_2>t\}} |\nabla u_{bl}|^2\,\mathrm{d}y$$

for $t > 1$. Indeed, from Lemma 2.1, there exists M such that $|Du_{bl}| \leq M$ for all y with $y_2 > 1$. In particular, in the case $p < 2$, when localized above $y_2 = 1$, the energy functional that appears at the left hand-side of (15) only involves the L^2 norm of the symmetric gradient (or of the gradient by the homogeneous Korn inequality, *cf* the appendix). Hence, $\nabla u_{bl} \in L^2(\Omega_{bl} \cap \{y_2 > 1\})$. The same holds for $p \geq 2$, thanks to (18).

Proposition 2.1 *There exists $C, \delta > 0$, such that $E(t) \leq C\exp(-\delta t)$.*

Proof. Let $t > 1$, $\Omega_{bl}^t := \Omega_{bl} \cap \{y_2 > t\}$. Let M such that $|Du_{bl}|$ is bounded by M over Ω_{bl}^1, which exists due to Lemma 2.1. As explained just above, one has $\int_{\Omega_{bl}^1} |Du_{bl}|^2 < +\infty$, and by Korn inequality $E(1)$ is finite. In particular, $E(t)$ goes to zero as $t \to +\infty$ and the point is to quantify the speed of convergence. By the use of inequality (13) (with $a = A$, $b = A + Du_{bl}$), we find

$$
\begin{aligned}
E(t) \le C \int_{\Omega_{bl}^t} |Du_{bl}|^2 \,\mathrm{d}y &\le C' \int_{\Omega_{bl}^t} \big(|A + Du_{bl}|^{p-2}(A + Du_{bl}) - |A|^{p-2}A\big) : Du_{bl} \,\mathrm{d}y \\
&\le C' \lim_{n\to\infty} \int_{\Omega_{bl}} \big(|A + Du_{bl}|^{p-2}(A + Du_{bl}) - |A|^{p-2}A\big) : Du_{bl}\, \chi_n(y_2) \,\mathrm{d}y
\end{aligned}
\tag{24}
$$

for a smooth χ_n with values in $[0, 1]$ such that $\chi_n = 1$ over $[t, t + n]$, $\chi_n = 0$ outside $[t - 1, t + n + 1]$, and $|\chi_n'| \le 2$. Then, we integrate by parts the right-hand side, taking into account the first equation in (8). We write

$$
\begin{aligned}
&\int_{\Omega_{bl}} \big(|A + Du_{bl}|^{p-2}(A + Du_{bl}) - |A|^{p-2}A\big) : Du_{bl}\, \chi_n(y_2) \,\mathrm{d}y \\
&= -\int_{\Omega_{bl}} \nabla p_{bl} \cdot u_{bl} \chi_n(y_2) \,\mathrm{d}y - \int_{\Omega_{bl}} \Big((S(A + Du_{bl}) - S(A)) \begin{pmatrix} 0 \\ \chi_n' \end{pmatrix}\Big) \cdot u_{bl} \,\mathrm{d}y \\
&= \int_{\Omega_{bl}} \Big(S(A) - S(A + Du_{bl})) \begin{pmatrix} 0 \\ \chi_n' \end{pmatrix}\Big) \cdot u_{bl} \,\mathrm{d}y + \int_{\Omega_{bl}} p_{bl} \chi_n' u_{bl,2} \,\mathrm{d}y \;:=\; I_1 + I_2.
\end{aligned}
\tag{25}
$$

To estimate I_1 and I_2, we shall make use of simple vector inequalities. Namely:

$$
\text{for all } p \in]1, 2], \text{ for all vectors } a, b,\ a \neq 0, ,\quad \big||b|^{p-2}b - |a|^{p-2}a\big| \;\le\; C_{p,a}\, |b - a|,
\tag{26}
$$

whereas

$$
\text{for all } p > 2, \text{ for all vectors } a, b,\ |b| \le M,\quad \big||b|^{p-2}b - |a|^{p-2}a\big| \;\le\; C_{p,a,M}\, |b - a|.
\tag{27}
$$

The latter is a simple application of the finite increment inequality. As regards the former, we distinguish between two cases:

- If $|b - a| < \frac{|a|}{2}$, it follows from the finite increments inequality.
- If $|b - a| \ge \frac{|a|}{2}$, we simply write

  $$\big||b|^{p-2}b - |a|^{p-2}a\big| \;\le\; |b|^{p-1} + |a|^{p-1} \;\le\; (3^{p-1} + 2^{p-1})|b - a|^{p-1} \;\le\; (3^{p-1} + 2^{p-1})(\frac{|a|}{2})^{p-2}|b - a|$$

  using that $\left(\frac{2|b-a|}{|a|}\right)^{p-1} \le \left(\frac{2|b-a|}{|a|}\right)$ for $1 < p \le 2$.

We shall also make use of the following:

Lemma 2.2 *For any height $t > 0$*

i) $\int_{\{y_2=t\}} u_{bl,2} = 0$.

ii) $\int_{\{y_2=t\}} (S(A + Du_{bl}) - S(A)) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$.

Proof of the lemma.

i) The integration of the divergence-free condition over $\Omega_{bl}^{0,t}$ leads to

$$
\begin{aligned}
0 = \int_{\Omega_{bl}^{0,t}} \operatorname{div} u_{bl} &= \int_{\{y_2=t\}} u_{bl,2} - \int_{\{y_2=0^+\}} u_{bl,2} = \int_{\{y_2=t\}} u_{bl,2} - \int_{\{y_2=0^-\}} u_{bl,2} \\
&= \int_{\{y_2=t\}} u_{bl,2} - \int_{\Omega_{bl-}} \operatorname{div} u_{bl} + \int_{\Gamma_{bl}} u_{bl} \cdot n = \int_{\{y_2=t\}} u_{bl,2},
\end{aligned}
$$

where the second and fourth inequalities come respectively from the no-jump condition of u_{bl} at $y_2 = 0$ and the Dirichlet condition at Γ_{bl}.
ii) By integration of the first equation in (8) over $\Omega_{bl}^{0,t}$ we get:

$$\int_{y_2=t} (S(A + Du_{bl}) - S(A) - p_{bl}Id) \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) = \int_{y_2=0^+} (S(A + Du_{bl}) - S(A) - p_{bl}Id) \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right).$$

In particular, the quantity

$$I := \int_{y_2=t} (S(A + Du_{bl}) - S(A) - p_{bl}Id) \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right) = \int_{y_2=t} (S(A + Du_{bl}) - S(A)) \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)$$

is independent of the variable t. To show that it is zero, we apply inequality (26) or (27) with $a = A$ and $b = A + Du_{bl}$, so that

$$I^2 \,\leq\, C \left(\int_{\{y_2=t\}} |Du_{bl}| \right)^2 \,\leq\, C' \int_{\{y_2=t\}} |Du_{bl}|^2$$

(C is bounded by Lemma 2.1). As Du_{bl} belongs to $L^2(\Omega_{bl}^1)$, there exists a sequence t_n such that $\int_{\{y_2=t_n\}} |Du_{bl}|^2 \to 0$ as $n \to +\infty$. It follows that $I = 0$. This concludes the proof of the Lemma.

We can now come back to the treatment of I_1 and I_2.

- Treatment of I_1.

We note that χ_n' is supported in $[t-1, t] \cup [t+n, t+n+1]$. By Lemma 2.2 we can write

$$I_1 = \int_{\Omega_{bl}^{t-1,t}} \Big(S(A) - S(A + Du_{bl})) \left(\begin{smallmatrix} 0 \\ \chi_n' \end{smallmatrix}\right) \Big) \cdot (u_{bl} - \overline{c}) \tag{28}$$

$$+ \int_{\Omega_{bl}^{t+n,t+n+1}} \Big(S(A) - S(A + Du_{bl})) \left(\begin{smallmatrix} 0 \\ \chi_n' \end{smallmatrix}\right) \Big) \cdot (u_{bl} - \overline{c}_n) \,:= I_{1,1} + I_{1,2}, \tag{29}$$

where

$$\overline{c} := \fint_{\Omega_{bl}^{t-1,t}} u_{bl} = \left(\fint_{\Omega_{bl}^{t-1,t}} u_{bl,1}, 0 \right) \quad \text{and} \quad \overline{c}_n := \fint_{\Omega_{bl}^{t+n,t+n+1}} u_{bl} = \left(\fint_{\Omega_{bl}^{t+n,t+n+1}} u_{bl,1}, 0 \right). \tag{30}$$

Again, we apply inequality (26) or (27) to find

$$I_{1,1} \leq C \int_{\Omega_{bl}^{t-1,t}} |Du_{bl}|\, |u_{bl} - \overline{c}|$$

and by the Poincaré inequality for functions with zero mean, we easily deduce that

$$I_{1,1} \leq C' \int_{\Omega_{bl}^{t-1,t}} |\nabla u_{bl}|^2 = C' \left(E(t-1) - E(t) \right).$$

An upper bound on $I_{1,2}$ can be derived in the same way:

$$I_{1,2} \leq C'(E(t+n) - E(t+n+1)),$$

where the right-hand side going to zero as $n \to +\infty$ since $E(t') \to 0$ as $t' \to \infty$. Eventually:

$$\limsup_{n\to+\infty} I_1 \,\leq\, C \left(E(t-1) - E(t) \right). \tag{31}$$

- Treatment of I_2.

We can again use the decomposition

$$I_2 = \int_{\Omega_{bl}^{t-1,t}} p_{bl}\chi_n' u_{bl,2} \; + \; \int_{\Omega_{bl}^{t+n,t+n+1}} p_{bl}\chi_n' u_{bl,2} \; := \; I_{2,1} + I_{2,2}. \tag{32}$$

From Lemma 2.2 i), we infer that

$$\int_{\Omega_{bl}^{t-1,t}} \chi_n'(y_2) u_{bl,2}(y) \, \mathrm{d}y = 0.$$

By standard results, there exists $w \in H_0^1(\Omega_{bl}^{t-1,t})$ satisfying $\mathrm{div}\, w(y) = \chi_n'(y_2)u_{bl,2}(y), \quad y \in \Omega_{bl}^{t-1,t}$, and the estimate

$$\|w\|_{H^1(\Omega_{bl}^{t-1,t})} \le C\|\chi_n'(y_2)u_{bl,2}(y)\|_{L^2(\Omega_{bl}^{t-1,t})} \le C'\|u_{bl,2}\|_{L^2(\Omega_{bl}^{t-1,t})},$$

for constants C, C' that do not depend on t. As w is zero at the boundary:

$$I_{1,2} = \int_{\Omega_{bl}^{t-1,t}} p_{bl} \mathrm{div}\, w = -\int_{\Omega_{bl}^{t-1,t}} \nabla p_{bl} \cdot w = \int_{\Omega_{bl}^{t-1,t}} (S(A + Du_{bl}) - S(A)) \cdot \nabla w$$

where the last equality comes from (8). We find as before (*cf* (26) or (27)):

$$\begin{aligned} |I_{1,2}| &\le C \int_{\Omega_{bl}^{t-1,t}} |Du_{bl}||\nabla w| \le C\|Du_{bl}\|_{L^2(\Omega_{bl}^{t-1,t})}\|\nabla w\|_{L^2(\Omega_{bl}^{t-1,t})} \\ &\le C'\|Du_{bl}\|_{L^2(\Omega_{bl}^{t-1,t})}\|u_{bl,2}\|_{L^2(\Omega_{bl}^{t-1,t})} \le C''\|\nabla u_{bl}\|^2_{L^2(\Omega_{bl}^{t-1,t})} \end{aligned}$$

where we have controlled the L^2 norm of $u_{bl,2}$ by the L^2 norm of its gradient (we recall that $u_{bl,2}$ has zero mean). A similar treatment can be performed with $I_{2,2}$, so that $I_{2,1} \le C(E(t-1) - E(t))$, $I_{2,2} \le C(E(t+n) - E(t+n+1))$ and

$$\limsup_{n\to+\infty} I_2 \; \le \; C\left(E(t-1) - E(t)\right). \tag{33}$$

Finally, combining (24), (25), (31) and (33), we get

$$E(t) \le C(E(t-1) - E(t))$$

for some $C > 0$. It is well-known that this kind of differential inequality implies the exponential decay of Proposition 2.1 (see the appendix). The proof of the Proposition is therefore complete. We have now all the ingredients to show Theorem 2.2.

Proof of Theorem 2.2. Thanks to the regularity Lemma 2.1, we know that ∇u_{bl} is uniformly bounded over Ω_{bl}^1, and belongs to $L^2(\Omega_{bl}^1)$. Of course, this implies that ∇u_{bl} belongs to $L^q(\Omega_{bl}^1)$ for all $q \in [2, +\infty]$. More precisely, combining the L^∞ bound with the L^2 exponential decay of Proposition 2.1, we have that

$$\|\nabla u_{bl}\|_{L^q(\Omega_{bl}^t)} \le C\exp(-\delta t) \tag{34}$$

(for some C and δ depending on q). This exponential decay extends straightforwardly to all $1 \le q < +\infty$. Let us now fix $q > 2$. To understand the behavior of u itself, we write the Sobolev inequality: for all y and $y' \in B(y, r)$,

$$|u(y') - u(y)| \; \le \; Cr^{1-\frac{2}{q}} \left(\int_{B(y,2r)} |\nabla u(z)|^q dz \right)^{1/q}. \tag{35}$$

We deduce from there that: for all $y_2 > 2$, for all $s \geq 0$,

$$
\begin{aligned}
&|u_{bl}(y_1, y_2 + s) - u_{bl}(y_1, y_2)| \\
&\leq |u_{bl}(y_1, y_2 + s) - u_{bl}(y_1, y_2 + \lfloor s \rfloor)| + \sum_{k=0}^{\lfloor s \rfloor - 1} |u_{bl}(y_1, y_2 + k + 1) - u_{bl}(y_1, y_2 + k)| \\
&\leq C \left(\|\nabla u_{bl}\|_{L^q(B((y_1, y_2+s)^t, 1))} + \sum_{k=0}^{\lfloor s \rfloor - 1} \|\nabla u_{bl}\|_{L^q(B((y_1, y_2+k)^t, 1))} \right) \\
&\leq C' \left(e^{-\delta(y_2+s)} + \sum_{k=0}^{\lfloor s \rfloor - 1} e^{-\delta(y_2+k)} \right)
\end{aligned}
$$

where the last inequality comes from (35). This implies that u_{bl} satisfies the Cauchy criterion uniformly in y_1, and thus converges uniformly in y_1 to some $u^\infty = u^\infty(y_1)$ as $y_2 \to +\infty$. To show that u^∞ is a constant field, we rely again on (35), which yields for all $|y_1 - y_1'| \leq 1$:

$$|u_{bl}(y_1, y_2) - u_{bl}(y_1', y_2)| \leq C|y_1 - y_1'|^{1-\frac{2}{q}} \|\nabla u_{bl}\|_{L^q(B((y_1, y_2)^t, 1))} \leq C' e^{-\delta y_2}.$$

Sending y_2 to infinity gives: $u^\infty(y_1) = u^\infty(y_1')$. Finally, the fact that u^∞ is a horizontal vector field follows from Lemma 2.2, point i). This concludes the proof of the Theorem 2.2.

Eventually, for later purposes, we state

Corollary 2.1 (higher order exponential decay)

- *There exists $\alpha \in (0,1)$, such that for all $s \in [0, \alpha)$, for all $1 \leq q < +\infty$, one can find C and $\delta > 0$ with*

$$\|u_{bl} - u^\infty\|_{W^{s+1,q}(\Omega_{bl}^t)} \leq C \exp(-\delta t), \quad \forall t \geq 1.$$

- *There exists $\alpha \in (0,1)$, such that for all $s \in [0, \alpha)$, for all $1 \leq q < +\infty$, one can find C and $\delta > 0$ with*

$$\|p_{bl} - p^t\|_{W^{s,q}(\Omega_{bl}^{t,t+1})} \leq C \exp(-\delta t), \quad \forall t \geq 1, \quad \textit{for some constant } p^t.$$

Proof of the corollary. It was established above that

$$|u(y_1, y_2 + s) - u(y_1, y_2)| \leq C' \left(e^{-\delta'(y_2+s)} + \sum_{k=0}^{\lfloor s \rfloor - 1} e^{-\delta'(y_2+k)} \right).$$

for some C' and $\delta' > 0$. From there, after sending s to infinity, it is easily deduced that

$$\|u_{bl} - u^\infty\|_{L^q(\Omega_{bl}^t)} \leq C \exp(-\delta t).$$

It then remains to control the $W^{s,q}$ norm of ∇u_{bl}. This control comes from the $C^{0,\alpha}$ uniform bound on ∇u_{bl} over Ω_{bl}^1, see Lemma 2.1. By Sobolev imbedding, it follows that

$$\|\nabla u_{bl}\|_{W^{s,q}(\Omega_{bl}^{t,t+1})} \leq C, \quad \forall s \in [0, \alpha), \forall 1 \leq q < +\infty$$

uniformly in t. Interpolating this bound with the bound $\|\nabla u_{bl}\|_{L^q(\Omega_{bl}^{t,t+1})} \leq C' \exp(-\delta' t)$ previously seen, we get

$$\|\nabla u_{bl}\|_{W^{s,q}(\Omega_{bl}^{t,t+1})} \leq C'' \exp(-\delta'' t), \quad \forall s \in [0, \alpha), \forall 1 \leq q < +\infty.$$

The first inequality of the Lemma follows.

The second inequality, on the pressure p_{bl}, is derived from the one on u_{bl}. This derivation is somehow standard, and we do not detail it for the sake of brevity.

3 Error estimates, wall Laws

3.1 Approximation by the Poiseuille flow.

We now go back to our primitive system (2). A standard estimate on u^ε leads to

$$\int_{\Omega^\varepsilon} |Du^\varepsilon|^p \;\le\; \int_{\Omega^\varepsilon} e_1 \cdot u^\varepsilon.$$

The Korn inequality implies that

$$\int_{\Omega^\varepsilon} |\nabla u^\varepsilon|^p \;\le\; C \int_{\Omega^\varepsilon} |Du^\varepsilon|^p$$

for a constant C independent of ε: indeed, one can extend u^ε by 0 for $x_2 < \varepsilon\gamma(x_1/\varepsilon)$ and apply the inequality on the square $\mathbb{T} \times [-1, 1]$, *cf* the appendix. Also, by the Poincaré inequality:

$$|\int_{\Omega^\varepsilon} e_1 \cdot u^\varepsilon| \le C\|u^\varepsilon\|_{L^p(\Omega^\varepsilon)} \le C'\|\nabla u^\varepsilon\|_{L^p(\Omega^\varepsilon)}.$$

We find that

$$\|u^\varepsilon\|_{W^{1,p}(\Omega^\varepsilon)} \le C. \tag{36}$$

In particular, it provides strong convergence of u^ε in L^p by the Rellich theorem (up to a subsequence). As can be easily guessed, the limit of u^0 in Ω is the generalized Poiseuille flow u^0. One can even obtain an error estimate by a direct energy estimate of the difference (extending u^0 and p^0 by zero in R^ε). We focus on the case $1 < p \le 2$, and comment briefly the easier case $p \ge 2$ afterwards. We write $u^\varepsilon = u^0 + w^\varepsilon$ and $p^\varepsilon = p^0 + q^\varepsilon$. We find, taking into account (5):

$$\begin{aligned}
-\mathrm{div}_x S(Du^0 + Dw^\varepsilon) + \mathrm{div}_x S(Du^0) + \nabla q^\varepsilon &= \mathbb{1}_{R^\varepsilon} e_1 \quad \text{ in } \Omega^\varepsilon \setminus \Sigma_0,\\
\mathrm{div}_x w^\varepsilon &= 0 \quad \text{ in } \Omega^\varepsilon,\\
w^\varepsilon &= 0 \quad \text{ on } \Gamma^\varepsilon \cup \Sigma_1,\\
w^\varepsilon & \text{ is periodic in } x_1 \text{ with period } 1,\\
[w^\varepsilon]|_{\Sigma_0} = 0, \quad [S(Du^0 + Dw^\varepsilon)n - S(Du^0)n - q^\varepsilon n]|_{\Sigma_0} &= -S(Du^0)n|_{x_2=0^+}.
\end{aligned} \tag{37}$$

In particular, performing an energy estimate and distinguishing between Ω and R^ε, we find

$$\int_\Omega \big(S(Du^0 + Dw^\varepsilon) - S(Du^0)\big) : Dw^\varepsilon + \int_{R^\varepsilon} S(Dw^\varepsilon) : Dw^\varepsilon = -\int_{\Sigma_0} S(Du^0)n|_{x_2=0^+} \cdot w^\varepsilon \mathrm{d}S + \int_{R^\varepsilon} e_1 \cdot w^\varepsilon \tag{38}$$

Relying on inequalities (13)-(14), we get for any $M > \|Du^0\|_{L^\infty}$:

$$\begin{aligned}
\|Dw^\varepsilon\|^p_{L^p(\Omega\cap\{|Dw^\varepsilon|\ge M\})} + \|Dw^\varepsilon\|^2_{L^2(\Omega\cap\{|Dw^\varepsilon|\le M\})} + \|Dw^\varepsilon\|^p_{L^p(R^\varepsilon)}&\\
\le C\left(\left|\int_{\Sigma_0} S(Du^0)n \cdot w^\varepsilon \,\mathrm{d}S\right| + \left|\int_{R^\varepsilon} e_1 \cdot w^\varepsilon\right|\right)&
\end{aligned} \tag{39}$$

Then by the Hölder inequality and by Proposition 4.2 in the appendix, we have that

$$|\int_{R^\varepsilon} e_1 \cdot w^\varepsilon| \le \varepsilon^{\frac{p-1}{p}} \|w^\varepsilon\|_{L^p(R^\varepsilon)} \le C\varepsilon^{1+\frac{p-1}{p}} \|\nabla w^\varepsilon\|_{L^p(R^\varepsilon)}. \tag{40}$$

Next, since Du^0 is given explicitly and uniformly bounded, the Proposition 4.2 provides

$$|\int_{\Sigma_0} (S(Du^0)n|_{x_2=0^+} \cdot w^\varepsilon \,\mathrm{d}S| \le C\|w^\varepsilon\|_{L^p(\Sigma_0)} \le C'\varepsilon^{\frac{p-1}{p}} \|\nabla w^\varepsilon\|_{L^p(R^\varepsilon)}. \tag{41}$$

Note that, as w^ε is zero at the lower boundary of the channel, we can extend it by 0 below R^ε and apply Korn inequalities in a strip (see the appendix). We find

$$\|\nabla w^\varepsilon\|_{L^p(R^\varepsilon)} \le C\|Dw^\varepsilon\|_{L^p(R^\varepsilon)}$$

for some constant $C > 0$ independent of ε. Summarising, we get

$$\|Dw^\varepsilon\|^p_{L^p(\Omega\cap\{|Dw^\varepsilon|\ge M\})} + \|Dw^\varepsilon\|^2_{L^2(\Omega\cap\{|Dw^\varepsilon|\le M\})} + \|Dw^\varepsilon\|^p_{L^p(R^\varepsilon)} \le C\varepsilon^{\frac{p-1}{p}}\|Dw^\varepsilon\|_{L^p(R^\varepsilon)}$$

and consequently

$$\|Dw^\varepsilon\|^p_{L^p(\Omega\cap\{|Dw^\varepsilon|\ge M\})} + \|Dw^\varepsilon\|^2_{L^2(\Omega\cap\{|Dw^\varepsilon|\le M\})} + \|Dw^\varepsilon\|^p_{L^p(R^\varepsilon)} \le C\varepsilon \tag{42}$$

In the case $p \ge 2$ one needs to use (16) instead of (13)-(14) what yields

$$\|Dw^\varepsilon\|_{L^p(\Omega^\varepsilon)} \le C\varepsilon^{\frac{1}{p}}, \quad p \in [2, \infty). \tag{43}$$

3.2 Construction of a refined approximation

The aim of this section is to design a better approximation of the exact solution u^ε of (2). This approximation will of course involve the boundary layer profile u_{bl} studied in the previous section. Consequences of this approximation in terms of wall laws will be discussed in paragraph 3.4.

From the previous paragraph, we know that the Poiseuille flow u^0 is the limit of u^ε in $W^{1,p}(\Omega)$. However, the extension of u^0 by 0 in the rough part of the channel was responsible for a jump of the stress tensor at Σ_0. This jump was the main limitation of the error estimates (42)-(43), and the reason for the introduction of the boundary layer term u_{bl}. Hence, we hope to have a better approximation replacing u^0 by $u^0 + \varepsilon u_{bl}(\cdot/\varepsilon)$. Actually, one can still improve the approximation, accounting for the so-called boundary layer tail u^∞. More precisely, *in the Newtonian case*, a good idea is to replace u^0 by the solution $u^{0,\varepsilon}$ of the Couette problem:

$$-\Delta u^{0,\varepsilon} + \nabla p^{0,\varepsilon} = 0, \quad \operatorname{div} u^{0,\varepsilon} = 0, \quad u^{0,\varepsilon}|_{\Sigma_0} = \varepsilon u^\infty, \quad u^{0,\varepsilon}|_{x_2=1} = 0.$$

One then defines:

$$u^\varepsilon = u^{0,\varepsilon} + \varepsilon(u_{bl}(\cdot/\varepsilon) - u_\infty) + r^\varepsilon \ \text{ in } \Omega, \quad u^\varepsilon = \varepsilon u_{bl}(\cdot/\varepsilon) \ \text{ in } R^\varepsilon,$$

where r^ε is a small divergence-free remainder correcting the $O(\exp(-\delta/\varepsilon))$ trace of $u_{bl} - u^\infty$ at $\{x_2 = 1\}$.

However, for technical reasons, the above approximation is not so successful in our context, so that we need to modify it a little. We proceed as follows. Let N a large constant to be fixed later. We introduce:

$$\Omega^\varepsilon_N := \Omega^\varepsilon \cap \{x_2 > N\varepsilon|\ln\varepsilon|\}, \quad \Omega^\varepsilon_{0,N} = \Omega^\varepsilon \cap \{0 < x_2 < N\varepsilon|\ln\varepsilon|\}, \quad \text{and} \quad \Sigma_N = \Pi \times \{x_2 = N\varepsilon|\ln\varepsilon|\}.$$

First, we introduce the solution $u^{0,\varepsilon}$ of

$$\begin{cases} -\operatorname{div} S(Du^{0,\varepsilon}) + \nabla p^{0,\varepsilon} = e_1, \quad x \in \Omega^\varepsilon_N, \\ \operatorname{div} u^{0,\varepsilon} = 0, \quad x \in \Omega^\varepsilon_N, \\ u^{0,\varepsilon}|_{\Sigma_N} = \left(x \to \begin{pmatrix} U'(0)x_2 \\ 0 \end{pmatrix} + \varepsilon u^\infty\right)|_{\Sigma_N}, \\ u^{0,\varepsilon}|_{\{x_2=1\}} = 0. \end{cases} \tag{44}$$

As for the generalized Poiseuille flow, the pressure $p^{0,\varepsilon}$ is zero, and one has an explicit expression for $u^{0,\varepsilon} = (U^\varepsilon(x_2), 0)$. In particular, one can check that

$$U^\varepsilon(x_2) = \beta(\varepsilon) - \frac{(\sqrt{2})^{p'}}{p'}\left|\frac{1}{2} + \alpha(\varepsilon) - x_2\right|^{p'}, \tag{45}$$

where $\alpha(\varepsilon)$ satisfies the equation ($x_{2,N} := N\varepsilon|\ln\varepsilon|$)

$$-\frac{1}{p'}(\sqrt{2})^{p'}\left(\left|\frac{1}{2}+\alpha(\varepsilon)-x_{2,N}\right|^{p'}-\left|\frac{1}{2}-\alpha(\varepsilon)\right|^{p'}\right)=U'(0)x_{2,N}+\varepsilon U^{\infty} \tag{46}$$

and

$$\beta(\varepsilon)=\frac{(\sqrt{2})^{p'}}{p'}\left|\frac{1}{2}-\alpha(\varepsilon)\right|^{p'}.$$

By the Taylor expansion, we find that

$$\alpha(\varepsilon)=-\sqrt{2}^{p'-4}\varepsilon U^{\infty}+O(\varepsilon^2|\ln\varepsilon|^2). \tag{47}$$

This will be used later.

Then, we consider the Bogovski problem

$$\begin{cases} \operatorname{div} r^{\varepsilon}=0 \quad \text{in } \Omega_N^{\varepsilon}, \\ r^{\varepsilon}|_{\Sigma_N}=\varepsilon(u_{bl}(\cdot/\varepsilon)-u^{\infty})|_{\Sigma_N}, \\ r^{\varepsilon}|_{\{x_2=1\}}=0. \end{cases} \tag{48}$$

Since $u^{\infty}=(U^{\infty},0)$, note that

$$\int_{\Sigma_N}\varepsilon(u_{bl}(\cdot/\varepsilon)-u^{\infty})\cdot e_2=\int_{\Omega_N^{\varepsilon}\cup\overline{R^{\varepsilon}}}\operatorname{div}_y u_{bl}(\cdot/\varepsilon)=0.$$

Hence, the compatibility condition for solvability of (48) is fulfilled: there exists a solution r^{ε} satisfying

$$\|r^{\varepsilon}\|_{W^{1,p}(\Omega_N^{\varepsilon})}\le C\varepsilon\|u_{bl}(\cdot/\varepsilon)-u^{\infty}\|_{W^{1-\frac{1}{p},p}(\Sigma_N)}.$$

Using the first estimate of Corollary 2.1, we find

$$\|r^{\varepsilon}\|_{W^{1,p}(\Omega_N^{\varepsilon})}\le C\varepsilon^{\frac{1}{p}}\exp(-\delta N|\ln\varepsilon|). \tag{49}$$

Finally, we define the approximation $(u^{\varepsilon}_{app},p^{\varepsilon}_{app})$ by the formula

$$u^{\varepsilon}_{app}(x)=\begin{cases} u^{0,\varepsilon}(x)+r^{\varepsilon}(x) \quad x\in\Omega_N^{\varepsilon}, \\ \begin{pmatrix} U'(0)x_2 \\ 0 \end{pmatrix}+\varepsilon u_{bl}(x/\varepsilon), \quad x\in\Omega^{\varepsilon}_{0,N}, \\ \varepsilon u_{bl}(x/\varepsilon), \quad x\in R^{\varepsilon}, \end{cases} \tag{50}$$

whereas

$$p^{\varepsilon}_{app}(x)=\begin{cases} 0 \quad x\in\Omega_N^{\varepsilon}, \\ p_{bl}(x/\varepsilon) \qquad x\in\Omega^{\varepsilon}_{0,N}\cup R^{\varepsilon}. \end{cases} \tag{51}$$

With such a choice:

$$u^{\varepsilon}_{app}|_{\partial\Omega^{\varepsilon}}=0, \quad \operatorname{div} u^{\varepsilon}_{app}=0 \quad \text{over } \Omega_N^{\varepsilon}\cup\Omega^{\varepsilon}_{0,N}\cup R^{\varepsilon}.$$

Moreover, u^{ε}_{app} has zero jump at the interfaces Σ_0 and Σ_N:

$$[u^{\varepsilon}_{app}]|_{\Sigma_0}=0, \quad [u^{\varepsilon}_{app}]|_{\Sigma_N}=0.$$

Still, the stress tensor has a jump. More precisely, we find

$$\begin{aligned} &\left[S(Du^{\varepsilon}_{app})n-p^{\varepsilon}_{app}n\right]|_{\Sigma_0}=0, \\ &\left[S(Du^{\varepsilon}_{app})n-p^{\varepsilon}_{app}n\right]|_{\Sigma_N}=\left(S(Du^0_{\varepsilon}+Dr^{\varepsilon})|_{\{x_2=(N\varepsilon|\ln\varepsilon|)^+\}}-S(A+Du_{bl}(\cdot/\varepsilon))|_{\{x_2=(N\varepsilon|\ln\varepsilon|)^-\}}\right)e_2 \\ &\qquad -p_{bl}(\cdot/\varepsilon)|_{\{x_2=(N\varepsilon|\ln\varepsilon|)^-\}}e_2. \end{aligned} \tag{52}$$

The next step is to obtain error estimates on $u^{\varepsilon}-u^{\varepsilon}_{app}$.

3.3 Error estimates

We prove here:

Theorem 3.1 (Error estimates)

- *For $1 < p \le 2$, there exists C such that*

$$\|u^\varepsilon - u^\varepsilon_{app}\|_{W^{1,p}(\Omega^\varepsilon)} \le C(\varepsilon|\ln\varepsilon|)^{1+\frac{1}{p'}}.$$

- *For $p \ge 2$, there exists C such that*

$$\|u^\varepsilon - u^\varepsilon_{app}\|_{W^{1,p}(\Omega^\varepsilon)} \le C(\varepsilon|\ln\varepsilon|)^{\frac{1}{p-1}+\frac{1}{p}}.$$

Remark 3.1 *A more careful treatment would allow to get rid of the* ln *factor in the last estimate ($p \ge 2$). We do not detail this point here, as we prefer to provide a unified treatment. Also, we recall that the shear thinning case ($1 < p \le 2$) has a much broader range of applications. More comments will be made on the estimates in the last paragraph 3.4.*

Proof of the theorem. We write $v^\varepsilon = u^\varepsilon - u^\varepsilon_{app}$, $q^\varepsilon = p^\varepsilon - p^\varepsilon_{app}$. We start from the equation

$$-\mathrm{div}\, S(Du^\varepsilon) + \mathrm{div}\, S(Du^\varepsilon_{app}) + \nabla q^\varepsilon = e_1 + \mathrm{div}\, S(Du^\varepsilon_{app}) + \nabla p^\varepsilon_{app} := F^\varepsilon \tag{53}$$

satisfied in $\Omega^\varepsilon \setminus (\Sigma_0 \cup \Sigma_N)$. A quick computation shows that

$$F^\varepsilon = \begin{cases} \mathrm{div}\, S(Du^{0,\varepsilon} + Dr^\varepsilon) - S(Du^{0,\varepsilon}), & x \in \Omega^\varepsilon_N, \\ e_1, \quad x \in \Omega^\varepsilon_{0,N} \cup R^\varepsilon. \end{cases}$$

Defining

$$\langle F^\varepsilon, v^\varepsilon\rangle := \int_{\Omega^\varepsilon_N} F^\varepsilon \cdot v^\varepsilon + \int_{\Omega^\varepsilon_{0,N}} F^\varepsilon \cdot v^\varepsilon + \int_{R^\varepsilon} F^\varepsilon \cdot v^\varepsilon$$

we get:

$$|\langle F^\varepsilon, v^\varepsilon\rangle| \le \alpha_\varepsilon \|\nabla v^\varepsilon\|_{L^p(\Omega^\varepsilon_N)} + \beta_\varepsilon \|v^\varepsilon\|_{L^p(\Sigma_N)} + \|v^\varepsilon\|_{L^1(\Omega^\varepsilon \setminus \Omega^\varepsilon_N)}$$

where

$$\alpha_\varepsilon := \|S(Du^{0,\varepsilon} + Dr^\varepsilon) - S(Du^{0,\varepsilon})\|_{L^{p'}(\Omega^\varepsilon_N)}, \quad \beta_\varepsilon := \| \left(S(Du^{0,\varepsilon} + Dr^\varepsilon) - S(Du^{0,\varepsilon})\right) e_2\|_{L^{p'}(\Sigma_N)}.$$

We then use the inequalities

$$\begin{aligned} \|v^\varepsilon\|_{L^p(\Sigma_N)} \ &\le \ C(\varepsilon|\ln\varepsilon|)^{1/p'} \|\nabla v^\varepsilon\|_{L^p(\Omega^\varepsilon)}, \\ \|v^\varepsilon\|_{L^1(\Omega^\varepsilon\setminus\Omega^\varepsilon_N)} \le C\varepsilon^{\frac{1}{p'}} \|v^\varepsilon\|_{L^p(\Omega^\varepsilon\setminus\Omega^\varepsilon_N)} \ &\le \ C\varepsilon^{\frac{1}{p'}} (\varepsilon|\ln\varepsilon|) \|\nabla v^\varepsilon\|_{L^p(\Omega^\varepsilon)} \end{aligned} \tag{54}$$

(see the appendix for similar ones). We end up with

$$|\langle F^\varepsilon, v^\varepsilon\rangle| \le C \left(\alpha_\varepsilon + \beta_\varepsilon (\varepsilon|\ln\varepsilon|)^{1/p'} + \varepsilon^{\frac{1}{p'}} (\varepsilon|\ln\varepsilon|)\right) \|\nabla v^\varepsilon\|_{L^p(\Omega^\varepsilon)}. \tag{55}$$

Back to (53), after multiplication by v^ε and integration over Ω^ε, we find:

$$\begin{aligned} &\int_{\Omega^\varepsilon} \left(S(Du^\varepsilon) - S(Du^\varepsilon_{app})\right) : \nabla v^\varepsilon \\ &\le C \left(\alpha_\varepsilon + \beta_\varepsilon (\varepsilon|\ln\varepsilon|)^{1/p'} + \varepsilon^{\frac{1}{p'}} (\varepsilon|\ln\varepsilon|)\right) \|\nabla v^\varepsilon\|_{L^p(\Omega^\varepsilon)} + \int_{\Sigma_N} \left([S(Du^\varepsilon_{app}) e_2]|_{\Sigma_N} \cdot v^\varepsilon - [p^\varepsilon_{app}]|_{\Sigma_N} v^\varepsilon_2\right). \end{aligned}$$

Let $p^{\varepsilon,N}$ be a constant to be fixed later. As v^ε is divergence-free and zero at Γ^ε, its flux through Σ_N is zero: $\int_{\Sigma_N} v_2^\varepsilon = 0$. Hence, we can add $p^{\varepsilon,N}$ to the pressure jump $[p^\varepsilon_{app}]|_{\Sigma_N}$ without changing the surface integral. We get:

$$
\begin{aligned}
&\int_{\Omega^\varepsilon} \big(S(Du^\varepsilon) - S(Du^\varepsilon_{app})\big) : \nabla v^\varepsilon \\
&\le C\left(\alpha_\varepsilon + \beta_\varepsilon(\varepsilon|\ln\varepsilon|)^{1/p'} + \varepsilon^{\frac{1}{p'}}(\varepsilon|\ln\varepsilon|)\right)\|\nabla v^\varepsilon\|_{L^p(\Omega^\varepsilon)} + \int_{\Sigma_N} \left([S(Du^\varepsilon_{app})e_2]|_{\Sigma_N}\cdot v^\varepsilon - ([p^\varepsilon_{app}]|_{\Sigma_N} - p^{\varepsilon,N})v_2^\varepsilon\right) \\
&\le \left(\alpha_\varepsilon + \beta_\varepsilon(\varepsilon|\ln\varepsilon|)^{1/p'} + \varepsilon^{\frac{1}{p'}}(\varepsilon|\ln\varepsilon|)\right)\|\nabla v^\varepsilon\|_{L^p(\Omega^\varepsilon)} + \gamma_\varepsilon\|v^\varepsilon\|_{L^p(\Sigma_N)} \\
&\le C\left(\alpha_\varepsilon + (\beta_\varepsilon+\gamma^\varepsilon)(\varepsilon|\ln\varepsilon|)^{1/p'} + \varepsilon^{\frac{1}{p'}}(\varepsilon|\ln\varepsilon|)\right)\|\nabla v^\varepsilon\|_{L^p(\Omega^\varepsilon)},
\end{aligned}
\tag{56}
$$

where

$$\gamma_\varepsilon := \|[\big(S(Du^\varepsilon_{app})]|_{\Sigma_N} - ([p^\varepsilon_{app}]|_{\Sigma_N} - p^{\varepsilon,N}\big)\, e_2)\|_{L^{p'}(\Sigma_N)}.$$

Note that we used again the first bound in (54) to go from the third to the fourth inequality.

Lemma 3.1 *For N large enough, and a good choice of $p^{\varepsilon,N}$ there exists $C = C(N)$ such that*

$$\alpha_\varepsilon \le C\varepsilon^{10}, \quad \beta^\varepsilon \le C\varepsilon^{10}, \quad \gamma_\varepsilon \le C\varepsilon|\ln\varepsilon|.$$

Let us temporarily admit this lemma. Then, we can conclude the proof of the error estimates:

- In the case $1 \le p \le 2$, we rely on the inequality established in [18, Proposition 5.2]: for all $p \in]1,2]$, there exists c such that for all $u, u' \in W^{1,p}_0(\Omega^\varepsilon)$

  $$\int_{\Omega^\varepsilon} (S(Du) - S(Du'))\cdot\nabla(u-u') \ge c\frac{\|Du - Du'\|^2_{L^p(\Omega^\varepsilon)}}{(\|Du\|_{L^p(\Omega^\varepsilon)} + \|Du'\|_{L^p(\Omega^\varepsilon)})^{2-p}}$$

  We use this inequality with $u = u^\varepsilon$, $u' = u^\varepsilon_{app}$. With the estimate (36) and the Korn inequality in mind, we obtain

  $$\int_{\Omega^\varepsilon} \big(S(Du^\varepsilon) - S(Du^\varepsilon_{app})\big)\cdot\nabla v^\varepsilon \ge c\|\nabla v^\varepsilon\|^2_{L^p}.$$

  Combining this lower bound with the upper bounds on $\alpha_\varepsilon, \beta_\varepsilon, \gamma_\varepsilon$ given by the lemma, we deduce from (56) the first error estimate in Theorem 3.1.

- In the case $2 \le p$, we use the easier inequality

  $$\int_{\Omega^\varepsilon} (S(Du) - S(Du'))\cdot\nabla(u-u') \ge c\|Du - Du'\|^p_{L^p(\Omega^\varepsilon)},$$

  so that

  $$\int_{\Omega^\varepsilon} \big(S(Du^\varepsilon) - S(Du^\varepsilon_{app})\big)\cdot\nabla v^\varepsilon \ge c\|\nabla v^\varepsilon\|^p_{L^p(\Omega^\varepsilon)}.$$

  The second error estimate from Theorem 3.1 follows.

The final step is to establish the bounds of Lemma 3.1.

Bound on α_ε and β_ε. From Corollary 2.1 and the trace theorem, we deduce that

$$\|u_{bl}(\cdot/\varepsilon) - u^\infty\|_{W^{1+s-\frac{1}{q},q}(\{x_2=t\})} \le C\varepsilon^{\frac{1}{q}-s-1}\exp(-\delta t/\varepsilon) \tag{57}$$

for some $s < \alpha$ (where $\alpha \in (0,1)$) and any $q > \frac{1}{s}$. Let $q > \max(p', \frac{2}{s})$. The solution r^ε of (48) satisfies: $r^\varepsilon \in W^{1+s,q}(\Omega_N^\varepsilon)$ with

$$\|r^\varepsilon\|_{W^{1+s,q}(\Omega_N^\varepsilon)} \le C\varepsilon^{\frac{1}{q}-s}\exp(-N\delta|\ln\varepsilon|)$$

so that by Sobolev imbedding

$$\|Dr^\varepsilon\|_{L^\infty(\Sigma_N)} + \|Dr^\varepsilon\|_{L^q(\Sigma_N)} + \|Dr^\varepsilon\|_{L^\infty(\Omega_N^\varepsilon)} \le C\|Dr^\varepsilon\|_{W^{s,q}(\Omega_N^\varepsilon)} \le C\varepsilon^{\frac{1}{q}-s}\exp(-N\delta|\ln\varepsilon|) \tag{58}$$

This last inequality allows to evaluate β_ε. Indeed, for $x \in \Sigma_N$, $C \ge |Du^{0,\varepsilon}(x)| \ge c > 0$ uniformly in x. We can then use the upper bound (26) for $p < 2$, or (27) for $p \ge 2$, to obtain

$$\beta_\varepsilon \le C\|Dr^\varepsilon\|_{L^{p'}(\Sigma_N)} \le C\|Dr^\varepsilon\|_{L^q(\Sigma_N)} \le C'\varepsilon^{\frac{1}{q}-s}\exp(-N\delta|\ln\varepsilon|) \le C'\varepsilon^{10} \tag{59}$$

for N large enough.

To treat α_ε, we still have to pay attention to the cancellation of $Du^{0,\varepsilon}$. Indeed, from the explicit expression of $u^{0,\varepsilon}$, we know that there is some $x_2(\varepsilon) \sim \frac{1}{2}$ at which $Du^{0,\varepsilon}|_{x_2 = x_2(\varepsilon)} = 0$. Namely, we write

$$\begin{aligned}
&\int_{\Omega_N^\varepsilon} |S(Du^{0,\varepsilon} + Dr^\varepsilon) - S(Du^{0,\varepsilon})|^{p'} \\
&= \int_{\{x\in\Omega_N^\varepsilon,\, |x_2 - x_2(\varepsilon)| \le \varepsilon^{10p'}\}} |S(Du^{0,\varepsilon} + Dr^\varepsilon) - S(Du^{0,\varepsilon})|^{p'} + \int_{\{x\in\Omega_N^\varepsilon,\, |x_2 - x_2(\varepsilon)| \ge \varepsilon^{10p'}\}} |S(Du^{0,\varepsilon} + Dr^\varepsilon) - S(Du^{0,\varepsilon})|^{p'} \\
&:= I_1 + I_2.
\end{aligned}$$

The first integral is bounded by

$$I_1 \le C\int_{\{x\in\Omega_N^\varepsilon,\, |x_2 - x_2(\varepsilon)| \le \varepsilon^{10p'}\}} |Du^{0,\varepsilon}|^p + |Dr^\varepsilon|^p \le C\varepsilon^{10p'},$$

where we have used the uniform bound satisfied by $Du^{0,\varepsilon}$ and Dr^ε over Ω_N^ε, see (58). For the second integral, we can distinguish between $p < 2$ and $p \ge 2$. For $p < 2$, see (26) and its proof, we get

$$I_2 \le C\int_{\{x\in\Omega_N^\varepsilon,\, |x_2 - x_2(\varepsilon)| \ge \varepsilon^{10p'}\}} |Du^{0,\varepsilon}|^{(p-2)p'}|Dr^\varepsilon|^{p'} \le C'\varepsilon^{-M}\exp(-\delta' N|\ln\varepsilon|) \tag{60}$$

for some $M, C', \delta' > 0$, see (58). In the case $p \ge 2$, as $Du^{0,\varepsilon}$ and Dr^ε are uniformly bounded, we derive a similar inequality by (27). In both cases, taking N large enough, we obtain $I_2 \le C''\varepsilon^{10p'}$, to end up with $\alpha_\varepsilon \le C\varepsilon^{10}$.

Bound on γ_ε. We have

$$\begin{aligned}
\gamma_\varepsilon \le\ & \|\left(S(Du^{0,\varepsilon} + Dr^\varepsilon) - S(Du^{0,\varepsilon})\right)e_2\|_{L^{p'}(\Sigma_N)} + \|\left(S(Du^{0,\varepsilon}) - S(A)\right)e_2\|_{L^{p'}(\Sigma_N)} \\
&+ \|(S(A) - S(A + Du_{bl}(\cdot/\varepsilon)))e_2\|_{L^{p'}(\Sigma_N)} + \|p_{bl}(\cdot/\varepsilon) - p^{\varepsilon,N}\|_{L^{p'}(\Sigma_N)}.
\end{aligned}$$

The first term is β_ε, so $O(\varepsilon^{10})$ by previous calculations. The third term can be treated similarly to β_ε. As $A \neq 0$, (26) implies that

$$\|(S(A) - S(A + Du_{bl}(\cdot/\varepsilon)))e_2\|_{L^{p'}(\Sigma_N)} \le C\|Du_{bl}(\cdot/\varepsilon)\|_{L^{p'}(\Sigma_N)} \le C'\exp(-\delta' N\ln\varepsilon), \tag{61}$$

where the last inequality can be deduced from (57). It is again $O(\varepsilon^{10})$ for N large enough. For the second term of the right-hand side, we rely on the explicit expression of $u^{0,\varepsilon}$. On the basis of (45)-(47), we find that

$$D(u^{0,\varepsilon})|_{\Sigma_N} = A + O(\varepsilon|\ln\varepsilon|)$$

resulting in

$$\| \left(S(Du^{0,\varepsilon}) - S(A)\right) e_2\|_{L^{p'}(\Sigma_N)} \leq C\varepsilon|\ln\varepsilon|.$$

Finally, to handle the pressure term, we use the second term of Corollary 2.1, which implies

$$\|p_{bl} - p^t\|_{L^q(\{y_2=t\})} \ \leq \ C\exp(-\delta t) \quad \text{for some constant } p^t.$$

We take $t = N\ln\varepsilon$ and $p^{\varepsilon,N} = p^t$ to get

$$\|p_{bl}(\cdot/\varepsilon) - p^{\varepsilon,N}\|_{L^{p'}(\Sigma_N)} \leq C' \exp(-\delta' N|\ln\varepsilon|).$$

Taking N large enough, we can make this term neglectible, say $O(\varepsilon^{10})$. Gathering all contributions, we obtain $\gamma_\varepsilon \leq C\varepsilon|\ln\varepsilon|$ as stated.

3.4 Comment on possible wall laws

On the basis of the previous error estimates, we can now discuss the appropriate wall laws for a non-Newtonian flow above a rough wall. We focus here again on the shear thinning case ($1 < p \leq 2$).

We first notice that the field u^ε_{app} (see (50)) involves in a crucial way the solution $u^{0,\varepsilon}$ of (44). Indeed, we know from (49) that the contribution of r^ε in $W^{1,p}(\Omega^\varepsilon_N)$ is very small for N large enough. Hence, the error estimate of Theorem 3.1 implies that

$$\|u^\varepsilon - u^{0,\varepsilon}\|_{W^{1,p}(\Omega^\varepsilon_N)} = O((\varepsilon|\ln\varepsilon|)^{1+\frac{1}{p'}}).$$

In other words, away from the boundary layer, u^ε is well approximated by $u^{0,\varepsilon}$, with a power of ε strictly bigger than 1. Although such estimate is unlikely to be optimal, it is enough to emphasize the role of the boundary layer tail u^∞. Namely, the addition of the term εu^∞ in the Dirichlet condition for $u^{0,\varepsilon}$ (see the third line of (44)) allows to go beyond a $O(\varepsilon)$ error estimate. *A contrario*, the generalized Poiseuille flow u^0 leads to a $O(\varepsilon)$ error only (away from the boundary layer). Notably,

$$\|u^\varepsilon - u^0\|_{W^{1,p}(\Omega^\varepsilon_N)} \geq \|u^{0,\varepsilon} - u^0\|_{W^{1,p}(\Omega^\varepsilon_N)} - \|u^\varepsilon - u^{0,\varepsilon}\|_{W^{1,p}(\Omega^\varepsilon_N)} \geq c\varepsilon - o(\varepsilon) \geq c'\varepsilon, \tag{62}$$

where the lower bound for $u^{0,\varepsilon} - u^0$ is obtained using the explicit expressions.

Let us further notice that instead of considering $u^{0,\varepsilon}$, we could consider the solution u^0_ε of

$$\begin{cases} -\operatorname{div} S(u^0_\varepsilon)) + \nabla p^0_\varepsilon = e_1, \quad x \in \Omega^\varepsilon_N, \\ \operatorname{div} u^0_\varepsilon = 0, \quad x \in \Omega^\varepsilon_N, \\ u^0_\varepsilon|_{\Sigma_0} = \varepsilon u^\infty, \\ u^0_\varepsilon|_{\{x_2=1\}} = 0. \end{cases} \tag{63}$$

It reads $u^0_\varepsilon = (U_\varepsilon, 0)$ with

$$U_\varepsilon(x_2) = \beta'(\varepsilon) - \frac{(\sqrt{2})^{p'}}{p'}\left|\frac{1}{2} + \alpha'(\varepsilon) - x_2\right|^{p'}$$

for α' and β' satisfying

$$-\frac{1}{p'}(\sqrt{2})^{p'}\left(\left|\frac{1}{2} + \alpha'(\varepsilon)\right|^{p'} - \left|\frac{1}{2} - \alpha'(\varepsilon)\right|^{p'}\right) = \varepsilon u^\infty_1 \quad \text{and} \quad \beta'(\varepsilon) = \frac{(\sqrt{2})^{p'}}{p'}\left|\frac{1}{2} - \alpha'(\varepsilon)\right|^{p'}.$$

We can compare directly these expressions to (45)-(46) and deduce that

$$\|u^{0,\varepsilon} - u^0_\varepsilon\|_{W^{1,p}(\Omega^\varepsilon_N)} = O(\varepsilon|\ln\varepsilon|),$$

which in turn implies that

$$\|u^\varepsilon - u^0_\varepsilon\|_{W^{1,p}(\Omega^\varepsilon_N)} = O(\varepsilon|\ln\varepsilon|). \tag{64}$$

Hence, in view of (62) and (64), we distinguish between two approximations (outside the boundary layer):

- A crude approximation, involving the generalized Poiseuille flow u^0.
- A refined approximation, involving u^0_ε.

The first choice corresponds to the Dirichlet wall law $u|_{\Sigma_0} = 0$, and neglects the role of the roughness. The second choice takes it into account through the inhomogeneous Dirichlet condition: $u|_{\Sigma_0} = \varepsilon u^\infty = \varepsilon(U^\infty, 0)$. Note that this last boundary condition can be expressed as a wall law, although slightly abstract. Indeed, U^∞ can be seen as a function of the tangential shear $(D(u^0)n)_\tau|_{\Sigma_0} = \partial_2 u^0_1|_{\Sigma_0} = U'(0)$, through the mapping

$$U'(0) \rightarrow A := \begin{pmatrix} 0 & U'(0) \\ U'(0) & 0 \end{pmatrix} \rightarrow u_{bl} \text{ solution of (8)-(9)} \rightarrow U^\infty = \lim_{y_2 \to +\infty} u_{bl,1}.$$

Denoting by $\mathcal{F}$ this application, we write

$$(u^0_\varepsilon)_\tau|_{\Sigma_0} = \varepsilon \mathcal{F}((D(u^0)n)_\tau|_{\Sigma_0}) \approx \varepsilon \mathcal{F}((D(u^0_\varepsilon)n)_\tau|_{\Sigma_0})$$

whereas $(u^0_\varepsilon)_n = 0$. This provides the following refined wall law :

$$u_n|_{\Sigma_0} = 0, \quad u_\tau|_{\Sigma_0} = \varepsilon \mathcal{F}\big((D(u)n)_\tau|_{\Sigma_0}\big).$$

This wall law generalizes the Navier wall law derived in the Newtonian case, where $\mathcal{F}$ is simply linear. Of course, it is not very explicit as it involves the nonlinear system (8)-(9). More studies will be necessary to obtain qualitative properties of the function $\mathcal{F}$, leading to a more effective boundary condition.

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4 Appendix : A few functional inequalities

Proposition 4.1 (Korn inequality) *Let $S_a := \mathbb{T} \times (a, a+1)$, $a \in \mathbb{R}$. For all $1 < p < +\infty$, there exists $C > 0$ such that: for all $a \in \mathbb{R}$, for all $u \in W^{1,p}(S_a)$,*

$$\|\nabla u\|_{L^p(S_a)} \le C\|Du\|_{L^p(S_a)}. \tag{65}$$

Proof. Without loss of generality, we can show the inequality for $a = 0$: the independence of the constant C with respect to a follows from invariance by translation. Let us point out that the keypoint of the proposition is that the inequality is homogeneous. Indeed, it is well-known that the inhomogeneous Korn inequality

$$\|\nabla u\|_{L^p(S_0)} \le C' \left(\|Du\|_{L^p(S_0)} + \|u\|_{L^p(S_0)}\right) \tag{66}$$

holds. To prove the homogeneous one, we use reductio at absurdum : if (65) is wrong, there exists a sequence u_n in $W^{1,p}(S_0)$ such that

$$\|\nabla u_n\|_{L^p(S_0)} \ge n\|Du_n\|_{L^p(S_0)}. \tag{67}$$

Up to replace u_n by $u_n' := (u_n - \int_{S_0} u_n)/\|u_n\|_{L^p}$, we can further assume that

$$\|u_n\|_{L^p} = 1, \quad \int_{S_0} u_n = 0.$$

Combining (66) and (67), we deduce that $1 \ge \frac{n-C'}{C'}\|D(u_n)\|_{L^p}$ which shows that $D(u_n)$ converges to zero in L^p. Using again (66), we infer that (u_n) is bounded in $W^{1,p}$, so that up to a subsequence it converges weakly to some $u \in W^{1,p}$, with strong convergence in L^p by Rellich Theorem. We have in particular

$$\|u\|_{L^p} = \lim_n \|u_n\|_{L^p} = 1, \quad \int_{S_0} u = \lim_n \int_{S_0} u_n = 0. \tag{68}$$

Moreover, as $D(u_n)$ goes to zero, we get $D(u) = 0$. This implies that u must be a constant (dimension is 2), which makes the two statements of (68) contradictory.

Corollary 4.1 *Let $H_a := \mathbb{T} \times (a, +\infty)$. For all $1 < p < +\infty$, there exists $C > 0$ such that: for all $a \in \mathbb{R}$, for all $u \in W^{1,p}(H_a)$,*

$$\|\nabla u\|_{L^p(H_a)} \le C\|Du\|_{L^p(H_a)}.$$

Proof. From the previous inequality, we get for all $n \in \mathbb{N}$:

$$\int_{S_{a+n}} |\nabla u|^p \le C \int_{S_{a+n}} |Du|^p.$$

The result follows by summing over n.

Corollary 4.2 *Let $1 < p < +\infty$. There exists $C > 0$, such that for all $u \in W^{1,p}(\Omega_{bl}^-)$, resp. $u \in W^{1,p}(\Omega_{bl})$, satisfying $u|_{\Gamma_{bl}} = 0$, one has*

$$\|\nabla u\|_{L^p(\Omega_{bl}^-)} \le C\|Du\|_{L^p(\Omega_{bl}^-)}, \quad \textit{resp. } \|\nabla u\|_{L^p(\Omega_{bl})} \le C\|Du\|_{L^p(\Omega_{bl})}.$$

Proof. One can extend u by 0 for all y with $-1 < y_2 < \gamma(y_1)$, and apply the previous inequality on S_{-1}, resp. H_{-1}.

Proposition 4.2 (Rescaled trace and Poincaré inequalities) *Let $\varphi \in W^{1,p}(R^\varepsilon)$. We have*

$$\|\varphi\|_{L^p(\Sigma)} \le C\varepsilon^{\frac{1}{p'}}\|\nabla_x\varphi\|_{L^p(R_\varepsilon)}, \tag{69}$$

$$\|\varphi\|_{L^p(R_\varepsilon)} \le C\varepsilon\|\nabla_x\varphi\|_{L^p(R_\varepsilon)}. \tag{70}$$

Proof. Let $\tilde{\varphi}(y) = \varphi(\varepsilon y)$, where $y \in S_k = S + (k, -1)$ (a rescaled single cell of rough layer). Then $\tilde{\varphi} \in W^{1,p}(S_k)$ for all $k \in \mathbb{N}$, and $\varphi = 0$ on Γ. By the trace theorem and the Poincaré inequality: for all $p \in [1, \infty)$

$$\int_{S_k \cap \{y_2=0\}} |\tilde{\varphi}(\bar{y}, 0)|^p \, \mathrm{d}\bar{y} \leq C \int_{S_k} |\nabla_y \tilde{\varphi}|^p \, \mathrm{d}y.$$

A change of variables provides

$$\int_{\varepsilon S_k \cap \{x_2=0\}} |\varphi(\bar{x}, 0)|^p \varepsilon^{-1} \, \mathrm{d}\bar{x} \leq C \int_{\varepsilon S_k} \varepsilon^p |\nabla_x \tilde{\varphi}(x)|^p \varepsilon^{-2} \, \mathrm{d}x.$$

Summing over k we obtain

$$\left(\int_{\Sigma} |\varphi(\tilde{x}, 0)|^p \, \mathrm{d}\tilde{x} \right)^{\frac{1}{p}} \leq C \varepsilon^{\frac{p-1}{p}} \left(\int_{R^\varepsilon} |\nabla_x \varphi(x)|^p \, \mathrm{d}x \right)^{\frac{1}{p}}$$

and (69) is proved. The inequality (70) is proved in the same way, as a consequence of the (one-dimensional) Poincaré inequality. $\square$